

COMPLEMENTATION FOR RIGHT IDEALS IN GENERALIZED HILBERT ALGEBRAS

BY

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ABSTRACT. Let \mathfrak{A} be a generalized Hilbert algebra and let \mathfrak{J} be a closed right ideal of \mathfrak{A} . Let \mathfrak{J}^\perp denote the pre-Hilbert space orthogonal complement of \mathfrak{J} in \mathfrak{A} . The problem investigated in this paper is: for which algebras \mathfrak{A} is it true that $\mathfrak{A} = \mathfrak{J} \oplus \mathfrak{J}^\perp$ for every closed right ideal \mathfrak{J} of \mathfrak{A} ? In the case that \mathfrak{A} is achieved, a slightly stronger property is characterized and these characterizations are then used to investigate some interesting examples.

Introduction. It is known that for full Hilbert algebras, the complementation property holds for both closed left and right ideals. This fact is due to Yood [7]. For achieved generalized Hilbert algebras the problem is more subtle although not difficult. A number of characterizations of a strong complementation property are given below. Perhaps the most useful of these is Theorem 1.11. Several cases are analyzed using this theorem. I would like to thank T. W. Palmer for suggesting this problem to me.

Definitions and notation. The definitions and notation of M. Takesaki's printed notes [6] will usually be used without reference. In particular, if \mathfrak{U} is a generalized Hilbert algebra, its involution will be noted by $\#$ and for $\xi \in \mathfrak{U}$, $\pi(\xi)$ will denote the unique continuous extension to the completion of \mathfrak{U} of the operator "left multiplication by ξ ". If \flat is the conjugate linear adjoint of $\#$ and if \mathfrak{N} is in the domain of \flat then one can define an operator on \mathfrak{U} by: $\pi'(\mathfrak{N})\xi = \pi(\xi)\mathfrak{N}$ for all $\xi \in \mathfrak{U}$. If $\pi'(\mathfrak{N})$ is bounded on \mathfrak{U} then the unique bounded extension of $\pi'(\mathfrak{N})$ to the completion of \mathfrak{U} is denoted by $\pi'(\mathfrak{N})$ also. The set of all \mathfrak{N} in the domain of \flat such that $\pi'(\mathfrak{N})$ is bounded will be denoted by \mathfrak{U}' . By Lemma 3.3 of [6], \mathfrak{U}' is an algebra with involution \flat such that $(\mathfrak{U}')^2$ is dense in the domain of \flat . In particular, \mathfrak{U}' is dense in the completion, $\mathcal{H}(\mathfrak{U})$ of \mathfrak{U} . If ξ is any element of $\mathcal{H}(\mathfrak{U})$ one can define an operator $\pi(\xi)$ with domain \mathfrak{U}' by: $\pi(\xi)\mathfrak{N} = \pi'(\mathfrak{N})\xi$ for all $\mathfrak{N} \in \mathfrak{U}'$. If $\pi(\xi)$ is bounded on \mathfrak{U}' then the unique bounded extension of $\pi(\xi)$ to $\mathcal{H}(\mathfrak{U})$ is denoted by $\pi(\xi)$ also. The set of all elements ξ of $\mathcal{H}(\mathfrak{U})$ such that $\pi(\xi)$ is bounded will be denoted by \mathcal{H}_l .

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If \mathfrak{U} is a generalized Hilbert algebra, a subset K of $\mathcal{H}(\mathfrak{U})$ is said to be *right invariant*; *left invariant*; or *two-sided invariant* respectively if $\pi'(\mathfrak{U}')K \subseteq K$; $\pi(\mathfrak{U})K \subseteq K$; or both $\pi'(\mathfrak{U}')K \subseteq K$ and $\pi(\mathfrak{U})K \subseteq K$. If \mathcal{H} is any Hilbert space and K is any subset of \mathcal{H} then K^\oplus is defined by $K^\oplus = \{\xi \in \mathcal{H} \mid (\xi|\eta) = 0 \text{ for all } \eta \in K\}$. If \mathfrak{U} is any generalized Hilbert algebra and \mathcal{H} is the completion of \mathfrak{U} , then for any subset \mathcal{J} of \mathfrak{U} , \mathcal{J}^\perp is defined by $\mathcal{J}^\perp = \mathcal{J}^\oplus \cap \mathfrak{U}$. Moreover, $\overline{\mathcal{J}}^\mathcal{H}$ is defined to be the closure of \mathcal{J} in \mathcal{H} and $\overline{\mathcal{J}} = \overline{\mathcal{J}}^\mathcal{H} \cap \mathfrak{U}$ is the relative closure of \mathcal{J} in \mathfrak{U} .

1. The complementation property.

1.1 Lemma. *Let \mathfrak{U} be a generalized Hilbert algebra. If \mathcal{J} is a right (left; two-sided) ideal of \mathfrak{U} , then \mathcal{J}^\perp is a closed right (left; two-sided) ideal of \mathfrak{U} . Moreover, $\overline{\mathcal{J}}^\mathcal{H}$ and \mathcal{J}^\oplus are closed right (left; two-sided) invariant subspaces of $\mathcal{H}(\mathfrak{U})$.*

Proof. Since π is a *-representation of \mathfrak{U} as bounded operators on \mathcal{H} , the proof for left ideals is easy. Thus, it suffices to give the proof for right ideals.

Let \mathcal{J} be a right ideal of \mathfrak{U} . It is first shown that $\overline{\mathcal{J}}^\mathcal{H}$ is a right invariant subspace of \mathcal{H} . Let $\eta \in \mathfrak{U}'$ and let $\xi \in \overline{\mathcal{J}}^\mathcal{H}$. Let $\{\xi_n\} \subseteq \mathcal{J}$ and $\{\eta_n\} \subseteq \mathfrak{U}$ be sequences such that $\lim_n \xi_n = \xi$ and $\lim_n \eta_n = \eta$. Then,

$$\pi'(\eta)\xi = \lim_n \pi'(\eta)\xi_n = \lim_n \pi(\xi_n)\eta = \lim_n \lim_k \pi(\xi_n)\eta_k.$$

But, $\pi(\xi_n)\eta_k = \xi_n\eta_k \in \mathcal{J}$ so that $\pi'(\eta)\xi \in \overline{\mathcal{J}}^\mathcal{H}$. Therefore, $\overline{\mathcal{J}}^\mathcal{H}$ is right invariant and since π' is an anti-*-representation of \mathfrak{U}' on \mathcal{H} we have that \mathcal{J}^\oplus is also right invariant.

To see that \mathcal{J}^\perp is a right ideal of \mathfrak{U} , let $\zeta \in \mathcal{J}^\perp$ and let $\xi \in \mathfrak{U}$. Let $\{\xi_n\} \subseteq \mathfrak{U}'$ be a sequence such that $\lim_n \xi_n = \xi$. Then, $\zeta\xi \in \mathfrak{U}$ and $\zeta\xi = \lim_n \pi(\xi)\xi_n = \lim_n \pi'(\xi_n)\zeta$. But, $\pi'(\xi_n)\zeta \in \mathcal{J}^\oplus$ so that $\zeta\xi \in \mathcal{J}^\oplus \cap \mathfrak{U} = \mathcal{J}^\perp$.

1.2 Lemma. *Let \mathfrak{U} be a generalized Hilbert algebra such that $\pi(\mathfrak{U})$ is a left ideal in $L(\mathfrak{U})$. Let E be a closed right invariant subspace of $\mathcal{H}(\mathfrak{U})$. Then $E \cap \mathfrak{U}$ is dense in E and is moreover a closed right ideal of \mathfrak{U} .*

Proof. Let P be the orthogonal projection on E . Since P commutes with $\pi'(\mathfrak{U}')$, P is in $L(\mathfrak{U})$. Hence $\pi(P(\mathfrak{U})) = P\pi(\mathfrak{U}) \subseteq \pi(\mathfrak{U})$ or $P(\mathfrak{U}) \subseteq \mathfrak{U}$. Thus, $P(\mathfrak{U}) \subseteq \mathfrak{U} \cap E$ is dense in E since \mathfrak{U} is dense in \mathcal{H} . It is fairly clear that $\mathfrak{U} \cap E$ is a closed right ideal of \mathfrak{U} .

1.3 Definition. Let \mathfrak{U} be a generalized Hilbert algebra. Then, \mathfrak{U} is said to be *strongly orthocomplemented* if for every closed right ideal \mathcal{J} of \mathfrak{U} , it is true that $\mathfrak{U} = \mathcal{J} \oplus \mathcal{J}^\perp$, and for every nonzero projection P in $L(\mathfrak{U})$, $P(\mathfrak{U}) \cap \mathfrak{U} \neq \{0\}$.

1.4 Example. An achieved generalized Hilbert algebra \mathfrak{U} and a nonzero projection P in $L(\mathfrak{U})$ such that $P(\mathfrak{U}) \cap \mathfrak{U} = \{0\}$.

Let \mathcal{H} be a separable Hilbert space with orthonormal basis $\{x_n\}_{n=1}^\infty$.

Let $\mathcal{B}(\mathcal{H})$ denote the von Neumann algebra of all bounded linear operators on \mathcal{H} and define a function ϕ on the positive elements T of $\mathcal{B}(\mathcal{H})$, via $\phi(T) = \sum_{n=1}^{\infty} n(Tx_n | x_n)$. Then ϕ satisfies the hypotheses of Theorem 2.13 of [2] and so if $\mathfrak{U} = \{T \in \mathcal{B}(\mathcal{H}) | \phi(T^*T) + \phi(TT^*) < \infty\}$ then \mathfrak{U} is an achieved generalized Hilbert algebra with linear product $(T|S) = \phi(S^*T)$ (ϕ can be uniquely extended to \mathfrak{U}) and $L(\mathfrak{U})$ is naturally isomorphic to $\mathcal{B}(\mathcal{H})$. Let P be the projection on the vector $\sum_{n=1}^{\infty} n^{-1}x_n$. Then, $P \in \mathcal{B}(\mathcal{H})$ but, $P \notin \mathfrak{U}$ since $\phi(P) = \infty$. Now, $P(\mathfrak{U}) \cap \mathfrak{U}$ is a right ideal of \mathfrak{U} and so by Theorem 1.14 of this paper there is a nonzero projection $Q \in P(\mathfrak{U}) \cap \mathfrak{U}$ if $P(\mathfrak{U}) \cap \mathfrak{U} \neq \{0\}$. But, then $P(Q) = Q$ and so $Q \leq P$. However, P is a minimal projection of $L(\mathfrak{U})$ and so $Q = P$, a contradiction. Hence, $P(\mathfrak{U}) \cap \mathfrak{U} = \{0\}$. Virtually the same example is given on p. 59 of [2], although the above example was discovered independently.

1.5 Proposition. *Let \mathfrak{U} be a generalized Hilbert algebra such that $\pi(\mathfrak{U})$ is a left ideal in $L(\mathfrak{U})$. Then \mathfrak{U} is strongly orthocomplemented.*

Proof. Let \mathcal{J} be a closed right ideal of \mathfrak{U} . Then $\mathcal{H} = \overline{\mathcal{J}^{\mathcal{H}}} \oplus \mathcal{J}^{\perp}$. Let $\xi \in \mathfrak{U}$ with $\xi = \zeta + \eta$ where $\zeta \in \mathcal{J}^{\mathcal{H}}$ and $\eta \in \mathcal{J}^{\perp}$. Let P be the orthogonal projection on $\mathcal{J}^{\mathcal{H}}$. Then as in the previous lemma $P\xi = \zeta \in \mathfrak{U} \cap \overline{\mathcal{J}^{\mathcal{H}}} = \mathcal{J}$. Therefore, $\eta \in \mathfrak{U} \cap \mathcal{J}^{\perp} = \mathcal{J}^{\perp}$ and so $\mathfrak{U} = \mathcal{J} \oplus \mathcal{J}^{\perp}$.

Let P be a nonzero projection in $L(\mathcal{Q})$. Since $P(\mathfrak{U}) \subseteq \mathfrak{U}$ by hypothesis and $P(\mathfrak{U}) \neq \{0\}$, $P(\mathfrak{U}) \cap \mathfrak{U} \neq \{0\}$.

The previous arguments are only slight modifications of those in [7]. Thus, it is perhaps surprising that Proposition 1.5 has a converse.

1.6 Lemma. *Let \mathfrak{U} be a generalized Hilbert algebra which is strongly orthocomplemented. If P is any projection in $L(\mathfrak{U})$ then $P(\mathfrak{U}) \subseteq \mathfrak{U}$.*

Proof. It is first shown that $P(\mathfrak{U}) \cap \mathfrak{U}$ is dense in the range of P . Let $\mathcal{K} = \overline{P(\mathfrak{U}) \cap \mathfrak{U}^{\mathcal{H}}}$. The, since $P(\mathfrak{U}) \cap \mathfrak{U}$ is a right ideal of \mathfrak{U} , \mathcal{K} is a closed right invariant subspace of $\mathcal{H}(\mathfrak{U})$ by Lemma 1.1. If Q is the projection on \mathcal{K} then $Q \in L(\mathfrak{U})$ and $Q \leq P$. Let $P - Q = R$ a projection in $L(\mathfrak{U})$. If $R \neq 0$ then there is a nonzero element $\xi \in \mathfrak{U}$ such that $R\xi = \xi$. But, then $P\xi = \xi$ since $P \geq R$ and so $\xi \in \mathcal{K}$. Thus, $\xi = Q\xi = P\xi - R\xi = 0$, a contradiction. Hence, $P(\mathfrak{U}) \cap \mathfrak{U}$ is dense in the range of P .

Let $\mathcal{J} = P(\mathfrak{U}) \cap \mathfrak{U}$. Since \mathcal{J} is a closed right ideal of \mathfrak{U} one has $\mathfrak{U} = \mathcal{J} \oplus \mathcal{J}^{\perp}$. But since \mathcal{J} is dense in the range of P , \mathcal{J}^{\perp} is contained in the orthogonal complement of the range of P . Hence, $P(\mathfrak{U}) = P(\mathcal{J}) = \mathcal{J} \subseteq \mathfrak{U}$.

1.7 Theorem. *Let \mathfrak{U} be a generalized Hilbert algebra which is strongly orthocomplemented. Then $\mathfrak{U} = \mathfrak{U}_1 \oplus^{\perp} \mathfrak{U}_2$ where \mathfrak{U}_1 is a commutative strongly orthocomplemented Hilbert algebra and \mathfrak{U}_2 is a (strongly orthocomplemented) generalized Hilbert algebra such that $\pi(\mathfrak{U}_2)$ is a left ideal in $L(\mathfrak{U}_2)$.*

Proof. Let P be the largest abelian central projection in $L(\mathfrak{U})$. Since \mathfrak{U} is strongly orthocomplemented, it is easy to see that $\mathfrak{U}_1 = P(\mathfrak{U})$ and $\mathfrak{U}_2 = (1 - P)(\mathfrak{U})$ are closed two-sided ideals of \mathfrak{U} and $\mathfrak{U} = \mathfrak{U}_1 \oplus^\perp \mathfrak{U}_2$. It is not difficult to see that each of \mathfrak{U}_1 and \mathfrak{U}_2 is invariant under $\#$ and hence they are generalized Hilbert algebras. Clearly, \mathfrak{U}_1 and \mathfrak{U}_2 are strongly orthocomplemented and \mathfrak{U}_1 is commutative. To see that \mathfrak{U}_1 is a Hilbert algebra, let $\{\xi_\alpha\}_{\alpha \in A}$ be a left approximate identity in \mathfrak{U}_1 and let $\eta, \zeta \in \mathfrak{U}_1$. Then,

$$\begin{aligned} \eta|\zeta &= \lim_\alpha (\xi_\alpha \eta|\zeta) = \lim_\alpha \eta(\xi_\alpha|\zeta) = \lim_\alpha (\xi_\alpha|\eta^\# \zeta) = \lim_\alpha (\xi_\alpha|\zeta \eta^\#) \\ &= \lim_\alpha (\zeta^\# \xi_\alpha|\eta^\#) = \lim_\alpha (\xi_\alpha \zeta^\#|\eta^\#) = (\zeta^\#|\eta^\#). \end{aligned}$$

Hence \mathfrak{U}_1 is actually a Hilbert algebra. Now, let $Q \in L(\mathfrak{U}_2)$ be a projection. Then by Lemma 1.6, $Q(\mathfrak{U}_2) \subseteq \mathfrak{U}_2$. Now, one can identify $L(\mathfrak{U}_2)$ and $(1 - P)L(\mathfrak{U})$ so that $L(\mathfrak{U}_2)$ has no nonzero abelian direct summands. Therefore, by a theorem of Fillmore and Topping [3], $L(\mathfrak{U}_2)$ is generated algebraically by its projections. That is, $T(\mathfrak{U}_2) \subseteq \mathfrak{U}_2$ for all $T \in L(\mathfrak{U}_2)$ which implies that $\pi(\mathfrak{U}_2)$ is a left ideal in $L(\mathfrak{U}_2)$.

1.8 Corollary. *Let \mathfrak{U} be a generalized Hilbert algebra such that $L(\mathfrak{U})$ has no infinite-dimensional abelian direct summands. Then \mathfrak{U} is strongly orthocomplemented if and only if $\pi(\mathfrak{U})$ is a left ideal in $L(\mathfrak{U})$.*

1.9 Example. A commutative strongly orthocomplemented Hilbert algebra \mathfrak{U} such that $\pi(\mathfrak{U})$ is not a left ideal in $L(\mathfrak{U})$.

Let \mathfrak{U} be the algebra of all Lebesgue measurable functions on $[0, 1]$ which assume only finitely many values. Give \mathfrak{U} the usual pointwise operations and inner product. Then \mathfrak{U} is strongly orthocomplemented but $\pi(\mathfrak{U})$ is not an ideal in $L(\mathfrak{U}) = \pi(L^\infty[0, 1])$.

1.10 Theorem. *Let \mathfrak{U} be an achieved generalized Hilbert algebra. Then the following are equivalent:*

1. $\mathfrak{U} = \mathcal{H}_I$.
2. $\pi(\mathfrak{U})$ is a left ideal in $L(\mathfrak{U})$.
3. \mathfrak{U} is strongly orthocomplemented.

Proof. Condition 1 implies condition 2 since $\pi(\mathcal{H}_I)$ is a left ideal in $L(\mathfrak{U})$ by Lemma 2.3 of [2]. Condition 2 implies condition 3 is just Proposition 1.5.

To see that condition 2 implies condition 1, let $\xi \in \mathcal{H}_I$ and let $\pi(\xi) = U\eta$ be the polar decomposition of $\pi(\xi)$. Now U is in $L(\mathfrak{U})$ and so $T = U^*\pi(\xi) = \pi(U^*\xi)$ is in $\pi(\mathcal{H}_I)$. But, since T is selfadjoint, $T \in \pi(\mathfrak{U})$. That is, $T = \pi(\zeta)$ for some $\zeta \in \mathfrak{U}$. Thus, $\pi(\xi) = U\pi(\zeta)$ is in $\pi(\mathfrak{U})$ by hypothesis and therefore $\xi \in \mathfrak{U}$. That is, $\mathfrak{U} = \mathcal{H}_I$.

To see that condition 3 implies condition 2 let \mathfrak{U} be strongly orthocomplemented and let $\mathfrak{U} = \mathfrak{U}_1 \oplus^\perp \mathfrak{U}_2$ where \mathfrak{U}_1 is a commutative Hilbert algebra and $\pi(\mathfrak{U}_2)$ is a left ideal in $L(\mathfrak{U}_2)$ by Theorem 1.7. By Lemma 1 of [5], \mathfrak{U}_1 is a full Hilbert algebra and so $\pi(\mathfrak{U}_1)$ is an ideal in $L(\mathfrak{U}_1)$. Hence, $\pi(\mathfrak{U})$ is an ideal in $L(\mathfrak{U}) = L(\mathfrak{U}_1) \oplus L(\mathfrak{U}_2)$.

Clearly, \mathfrak{H}_l itself is always strongly orthocomplemented so that condition 1 is not too surprising. Next, a useful inequality is obtained which tells when an achieved algebra is strongly orthocomplemented.

1.11 Theorem. *Let \mathfrak{U} be an achieved generalized Hilbert algebra. Then \mathfrak{U} is strongly orthocomplemented if and only if there is a $K > 0$ such that for all $\xi \in \mathfrak{U}$, $\|\xi^\# \| \leq K(\|\xi\| + \|\pi(\xi)\|)$.*

Proof. Suppose that \mathfrak{U} is strongly orthocomplemented, then by Theorem 1.10, $\mathfrak{U} = \mathfrak{H}_l$. However, exactly as in Proposition 1.15 of [4], it is seen that $\xi \mapsto \|\xi\| + \|\xi^\# \| + \|\pi(\xi)\|$ defines a Banach algebra norm on \mathfrak{U} and $\xi \mapsto \|\xi\| + \|\pi(\xi)\|$ defines a Banach algebra norm on \mathfrak{H}_l . Since $\mathfrak{U} = \mathfrak{H}_l$, the open mapping theorem shows that there is a constant $M > 1$ such that for all ξ in \mathfrak{U} $\|\xi\| + \|\xi^\# \| + \|\pi(\xi)\| \leq M(\|\xi\| + \|\pi(\xi)\|)$. Therefore, $\|\xi^\# \| \leq (M-1)(\|\xi\| + \|\pi(\xi)\|)$ for all ξ in \mathfrak{U} .

On the other hand, suppose that the condition holds. Let $\zeta \in \mathfrak{H}_l$ and let $\pi(\zeta) = U\pi(\xi)$ be the polar decomposition of $\pi(\zeta)$. Now, as seen in the proof of Theorem 1.10, $\xi \in \mathfrak{U}$ and $\zeta = \xi^\#$. For each positive integer n let f_n be defined by

$$\begin{aligned} f_n(t) &= 1/t, \quad t \geq 1/n, \\ &= 0, \quad t < 1/n. \end{aligned}$$

Then, $f_n(\pi(\xi)) \in L(\mathfrak{U})$ and so $f_n(\pi(\xi))\xi \in \mathfrak{H}_l$. But, $\pi(f_n(\pi(\xi))\xi) = f_n(\pi(\xi))\pi(\xi)$ is a selfadjoint idempotent in $L(\mathfrak{U})$ and so $e_n = f_n(\pi(\xi))\xi \in \mathfrak{U}$. Since $\pi(\xi)$ is positive, $\{\pi(e_n)\}$ converges to the range projection of $\pi(\xi)$ in the strong operator topology. However, since \mathfrak{U}' contains an approximate identity, ξ is the closure of the range of $\pi(\xi)$ and so $\lim_n e_n \xi = \xi$. But, e_n and ξ commute so that $\lim_n \xi e_n = \xi$.

Now, let ϵ be any positive number and let e_n be such that $\|\xi e_n - \xi\| < \epsilon/2$. Then, since $\zeta = U\xi$, $\|\zeta e_n - \zeta\| = \|U\pi(\xi)e_n - U\xi\| \leq \|\xi e_n - \xi\| < \epsilon/2$. By the Kaplansky density theorem choose $\mathfrak{N} \in \mathfrak{U}$ such that $\|\pi(\mathfrak{N})\| \leq \|\pi(\zeta)\|$ and $\|\mathfrak{N}e_n - \zeta e_n\| < \epsilon/2$. Then, $\|\mathfrak{N}e_n - \zeta\| < \epsilon$ and $\|\pi(\mathfrak{N}e_n)\| \leq \|\pi(\mathfrak{N})\| \leq \|\pi(\zeta)\|$. The above argument implies that there is a sequence $\{\zeta_n\} \subseteq \mathfrak{U}$ such that $\lim_n \zeta_n = \zeta$ and $\|\pi(\zeta_n)\| \leq \|\pi(\zeta)\|$. By assumption this implies that $\{\zeta_n^\#\}$ is a bounded sequence and hence by Alaoglu's theorem there is a subsequence $\{\zeta_{n_k}^\#\}$ and an element μ in $\mathfrak{H}(\mathfrak{U})$ such that $\zeta_{n_k}^\#$ converges to μ in w^* -topology of $\mathfrak{H}(\mathfrak{U})$. Thus, for all \mathfrak{N} in \mathfrak{U}' ,

$$(\mu|\mathcal{H}) = \lim_k (\zeta_{n_k}^\#|\mathcal{H}) = \lim_k (\mathcal{H}^\flat|\zeta_{n_k}) = (\mathcal{H}^\flat|\zeta).$$

Therefore, ζ is in the domain of $\#$ and $\zeta^\# = \mu$. Thus, $\zeta \in \mathcal{U}$ and so $\mathcal{U} = \mathcal{H}_I$ which by Theorem 1.10 implies that \mathcal{U} is strongly orthocomplemented.

1.12 Remark. If \mathcal{U} is achieved and the involution is continuous on \mathcal{U} , then \mathcal{U} is strongly orthocomplemented by the previous theorem (there are easier ways to see this). If the involution is not necessarily continuous but \mathcal{U} has an identity 1, then it is easy to see that for every T in $L(\mathcal{U})$, $\pi(T(1)) = T$ so that $\pi(\mathcal{H}_I) = L(\mathcal{U})$. But, then $\pi(\mathcal{U}) = \pi(\mathcal{H}_I) \cap \pi(\mathcal{H}_I)^* = L(\mathcal{U})$ so that $\mathcal{U} = \mathcal{H}_I$ and therefore \mathcal{U} is strongly orthocomplemented by Theorem 1.10.

1.13 Tensor products. Let \mathcal{H} and \mathcal{K} be pre-Hilbert spaces and let $\mathcal{H} \otimes \mathcal{K}$ be the algebraic tensor product (over the complex numbers) of \mathcal{H} and \mathcal{K} . Then $\mathcal{H} \otimes \mathcal{K}$ has a pre-Hilbert space structure given by

$$\left(\sum_{k=1}^n x_k \otimes y_k \middle| \sum_{j=1}^m w_j \otimes z_j \right) = \sum_{k=1}^n \sum_{j=1}^m (x_k | w_j) (y_k | z_j).$$

Now, if \mathcal{U} and \mathcal{B} are generalized Hilbert algebras then $\mathcal{U} \otimes \mathcal{B}$ has a natural algebraic structure which makes $\mathcal{U} \otimes \mathcal{B}$ into an algebra with involution. To see that $\mathcal{U} \otimes \mathcal{B}$ is a generalized Hilbert algebra with the above pre-Hilbert space structure, it suffices to see that the involution on $\mathcal{U} \otimes \mathcal{B}$ is closable as a conjugate linear operator on $\mathcal{H}(\mathcal{U} \otimes \mathcal{B})$. However, it is easy to see that $\mathcal{H}(\mathcal{U} \otimes \mathcal{B})$ is just the completion of $\mathcal{H}(\mathcal{U}) \otimes \mathcal{H}(\mathcal{B})$ and so $\mathcal{U}' \otimes \mathcal{B}'$ is dense in $\mathcal{H}(\mathcal{U} \otimes \mathcal{B})$. Now, \flat (on $\mathcal{U}' \otimes \mathcal{B}'$) is a conjugate linear operator with domain dense in $\mathcal{H}(\mathcal{U} \otimes \mathcal{B})$ and so has an adjoint. Clearly, the adjoint of \flat extends the involution on $\mathcal{U} \otimes \mathcal{B}$ and since the adjoint of a densely defined conjugate linear operator is closed, the involution on $\mathcal{U} \otimes \mathcal{B}$ is closable.

The following question can now be posed: Suppose \mathcal{U} and \mathcal{B} are achieved generalized Hilbert algebras which are strongly orthocomplemented, is it then true that $(\mathcal{U} \otimes \mathcal{B})''$ is strongly orthocomplemented? Clearly, if both \mathcal{U} and \mathcal{B} have an identity (both have a continuous involution) then $(\mathcal{U} \otimes \mathcal{B})''$ will have an identity (continuous involution) so that by the previous remark $(\mathcal{U} \otimes \mathcal{B})''$ will be strongly orthocomplemented. Moreover, Proposition 1.13 will show that $(\mathcal{U} \otimes \mathcal{B})''$ can fail to be strongly orthocomplemented even if one chooses \mathcal{U} to have an identity and \mathcal{B} to be an H^* algebra. First, the following theorem (due essentially to Ambrose [1]) is needed. The proof here follows that of Rieffel [4].

1.14 Theorem. *Let \mathcal{U} be a generalized Hilbert algebra such that either \mathcal{U} is achieved or $\pi(\mathcal{U})$ is a left ideal in $L(\mathcal{U})$. Then any nonzero left or right ideal of \mathcal{U} contains a nonzero selfadjoint idempotent.*

Proof. The proof is given for right ideals only. Let \mathcal{I} be a right ideal of \mathcal{U} such that there is some $\xi \in \mathcal{I}$ with $\xi \neq 0$. Then, $\xi\xi^\# \in \mathcal{I}$ and $\xi\xi^\# \neq 0$. Let $\zeta = \xi\xi^\#$; then $\pi(\zeta)$ is a positive selfadjoint operator in $L(\mathcal{U})$. Let J be a closed interval of the real line whose interior meets the spectrum of $\pi(\zeta)$ but such that $0 \notin J$. Define

$$f(t) = 1/t^2, \quad \text{for } t \in J, \\ = 0, \quad \text{for } t \notin J.$$

Then, $T = f(\pi(\zeta)) \in L(\mathcal{U})$. Now, if \mathcal{U} is achieved then since $T\zeta \in \mathcal{H}_l$ and $\pi(T\zeta) = f(\pi(\zeta))\pi(\zeta)$ is a selfadjoint operator, $T\zeta \in \mathcal{U}$. On the other hand if $\pi(\mathcal{U})$ is a left ideal $L(\mathcal{U})$ then, clearly, $T\zeta \in \mathcal{U}$. Therefore, $\zeta T\zeta \in \mathcal{I}$, but, $\pi(\zeta T\zeta) = f(\pi(\zeta))\pi(\zeta)^2$ is a nonzero selfadjoint projection in $L(\mathcal{U})$ and so $\zeta T\zeta$ is a nonzero selfadjoint idempotent in \mathcal{I} .

1.15 Proposition. *Let \mathcal{U} and \mathcal{B} be achieved generalized Hilbert algebras which are strongly orthocomplemented. Suppose that \mathcal{U} has a discontinuous involution and that \mathcal{B} does not have an identity. Then $(\mathcal{U} \otimes \mathcal{B})''$ is not strongly orthocomplemented.*

Proof. We first show that the set of projections in \mathcal{B} is unbounded. Let \mathcal{E} be a maximal family of orthogonal projections in \mathcal{B} . Then, since \mathcal{B} is strongly orthocomplemented one can use the previous theorem and the maximality of \mathcal{E} to show that $\mathcal{B} = \overline{\sum_{e \in \mathcal{E}} e \mathcal{B}}$. Now, if the set of projections in \mathcal{B} were bounded, say, $K > 0$ then for every finite subset \mathcal{F} of \mathcal{E} ,

$$\sum_{e \in \mathcal{F}} \|e\|^2 = \left\| \sum_{e \in \mathcal{F}} e \right\|^2 \leq K^2.$$

That is, $\sum_{e \in \mathcal{E}} \|e\|^2$ would converge and so $\sum_{e \in \mathcal{E}} e$ would define a selfadjoint projection of \mathcal{B} . But, since $\sum_{e \in \mathcal{E}} e \mathcal{B} = \mathcal{B}$, $\sum_{e \in \mathcal{E}} e$ would be a left identity for \mathcal{B} and therefore an identity. Since \mathcal{B} has no identity by hypothesis, the set of projections in \mathcal{B} is unbounded.

Now, since the involution on \mathcal{U} is discontinuous, let $\{\xi_n\} \subseteq \mathcal{U}$ be a sequence such that $\lim_n \xi_n = 0$ but $\|\xi_n^\#\| = 1$ for all n . Now, choose a sequence of selfadjoint idempotents $\{e_n\} \subseteq \mathcal{B}$ such that $\|e_n\| > n\|\pi(\xi_n)\|$. If $(\mathcal{U} \otimes \mathcal{B})''$ were strongly orthocomplemented then by Theorem 1.11 there would exist a constant $K > 0$ such that $\|\zeta^\#\| \leq K(\|\zeta\| + \|\pi(\zeta)\|)$ for all ζ in $(\mathcal{U} \otimes \mathcal{B})''$. But, then $\|\xi_n^\# \otimes e_n\| \leq K(\|\xi_n \otimes e_n\| + \|\pi(\xi_n \otimes e_n)\|)$ or $0 < 1/K \leq \|\xi_n\| + \|\pi(\xi_n)\|/\|e_n\| < \|\xi_n\| + 1/n$, for all positive integers n which is a contradiction. Hence, $(\mathcal{U} \otimes \mathcal{B})''$ is not strongly orthocomplemented.

1.16 Direct sums. Let $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ be a collection of generalized Hilbert algebras. Then the algebraic direct sum of $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ is clearly a generalized Hilbert

algebra with the obvious operations and inner product. This algebra is denoted by $\sum_{\alpha \in A} \mathfrak{U}_\alpha$. Clearly $\sum_{\alpha \in A} \mathfrak{U}_\alpha$ is strongly orthocomplemented if and only if each \mathfrak{U}_α is strongly orthocomplemented. However, each \mathfrak{U}_α can be achieved and strongly orthocomplemented while $(\sum_{\alpha \in A} \mathfrak{U}_\alpha)''$ is not strongly orthocomplemented, as seen below.

1.17 Example. Let \mathcal{H} be a separable Hilbert space with orthonormal basis $\{x_n\}_{n=1}^\infty$. Let \mathfrak{U} be the set of all bounded linear operators on \mathcal{H} and let $\#$ denote the usual operator adjoint. Give \mathfrak{U} the following inner product:

$$(\xi|\eta) = \sum_{k=1}^{\infty} \frac{1}{2^k} (\xi x_k | \eta x_k) \quad \text{for } \xi, \eta \in \mathfrak{U}.$$

Then, \mathfrak{U} is an achieved generalized Hilbert algebra with identity and therefore is strongly orthocomplemented. Let $\mathfrak{B} = (\sum_{n=1}^\infty \mathfrak{U})''$. For each positive integer n , let $\xi_n \in \mathfrak{U}$ be defined by $\xi_n(z) = (z|x_n)x_1$ for all $z \in \mathcal{H}$. Then, for each n , $\|\xi_n\|^2 = 1/2^n$, but $\|\xi_n^\#\|^2 = 1/2$ so that $\sum_{n=1}^\infty \|\xi_n\|^2 < \infty$, but $\sum_{n=1}^\infty \|\xi_n^\#\|^2 = \infty$. Moreover $\|\pi(\xi_n)\| \leq 1$ for each n so that it is easy to see that $\{\xi_n\}_{n=1}^\infty \in \mathcal{H}_l$ but is not in \mathfrak{B} . By Theorem 1.10, \mathfrak{B} is not strongly orthocomplemented. One notes that \mathfrak{B} satisfies the second but not the first condition in the definition of strongly orthocomplemented. If one takes the direct sum of \mathfrak{B} with the example in 1.4, one obtains an achieved generalized Hilbert algebra which satisfies neither condition of the definition.

1.18 *The generalized Hilbert algebra of a group.* Let G be a locally compact group with left Haar measure μ and modular function Δ . Let $\mathfrak{U}_0(G)$ be the (convolution) algebra of all continuous complex valued functions on G with compact support. Equip $\mathfrak{U}_0(G)$ with the following inner product and involution:

$$(f|g) = \int_G f(x) \overline{g(x)} d\mu(x),$$

$$f^\#(y) = \Delta(y^{-1}) \overline{f(y^{-1})} \quad \text{for all } y \in G, \text{ and for all } f, g \in \mathfrak{U}_0(G).$$

Then, $\mathfrak{U}_0(G)$ is a generalized Hilbert algebra. Define $\mathfrak{U}(G) = (\mathfrak{U}_0(G))''$ and call $\mathfrak{U}(G)$ the generalized Hilbert algebra of G .

If G is unimodular, then $\mathfrak{U}(G)$ is a full Hilbert algebra and so is strongly orthocomplemented. On the other hand, if G is not unimodular then $\mathfrak{U}(G)$ is not strongly orthocomplemented. To see this, let G be unimodular so that Δ is unbounded on G . Since G is not discrete, for each positive integer n there is an open set $W_n \subseteq G$ with compact closure such that $\mu(W_n) \leq 1$ and $\Delta(x) > n^2$ for all $x \in W_n$. Now, the characteristic function χ_{W_n} of W_n is clearly in the domain of $\#$ and the operator $g \mapsto \chi_{W_n} * g$ is bounded for $g \in L^2(G) = \mathcal{H}(\mathfrak{U}(G))$ by $\mu(W_n) \leq 1$. Thus, $\chi_{W_n} \in \mathfrak{U}(G)$ for each positive integer n . Now, an easy calculation shows that

$$= \int_G \Delta(x) \chi_{W_n}(x) d\mu(x) \geq n^2 \mu(W_n).$$

If $\mathfrak{U}(G)$ were strongly orthocomplemented, then by Theorem 1.11 there would exist $K > 0$ such that for each n

$$n\mu(W_n)^{1/2} \leq \|\chi_{W_n}^\# \| \leq K(\|\chi_{W_n}\| + \|\pi(\chi_{W_n})\|) \leq K(\mu(W_n)^{1/2} + \mu(W_n)).$$

Or, $n \leq K(1 + \mu(W_n)^{1/2}) \leq 2K$ for all positive integers n which is a contradiction. Hence $\mathfrak{U}(G)$ is not strongly orthocomplemented.

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